a) $y' - 2y = 6$

b) $y' + y = 3e^{-t}$

c) $7y' - y = e^{2t} + 3$

d) $y' + 2ty = 1$

5. Find the value of y_0 so that the solution of the IVP is bounded as $t \to \infty$.

   a) $y' - y = 1 + 5\cos(2t)$, where $y(0) = y_0$

   b) $y' - 3y = \sin t + 3\cos t$, where $y(0) = y_0$

6. A Maxwell viscoelastic material is one for which the stress $T(t)$ and the strain-rate $r(t)$ satisfy

$$T + \tau \frac{dT}{dt} = \kappa r,$$

   where τ and κ are positive constants. By solving this equation for T, and assuming $T_0 = T(0)$, show that

$$T = T_0 e^{-t/\tau} + \frac{\kappa}{\tau} \int_0^t e^{(s-t)/\tau} r(s) ds.$$

7. The Bernoulli equation is $w' = p(t)w + q(t)w^n$, which is nonlinear if $n \neq 0, 1$. What is significant is that it can be solved by making the substitution $w = y^{1/(1-n)}$, which results in a linear equation for $y(t)$. This was discovered by Leibniz, although it is not clear he was aware of the solution (2.21) for a linear equation [Parker, 2013].

   a) If $w' = w - 5w^3$, where $w(0) = 1$, what IVP does y satisfy?

   b) Solve the IVP for y, and then transform back to determine the function w.

   c) One of Bernoulli's brothers solved the Bernoulli equation by assuming that $w(t) = u(t)v(t)$, where u satisfies $u' = pu$, for $u(0) = 1$. Use this method to solve $w' = w - 5w^3$, where $w(0) = 1$. This approach is the precursor to what is now known as the method of variation of parameters.

2.3 ▪ Modeling

The principal objective of the examples to follow is to show how a differential equation is the mathematical consequence of the assumptions about a physical system.

2.3.1 ▪ Mixing

Typical mixing problems involve a continuously stirred tank, as illustrated in Figure 2.2. As an example, suppose that water, containing salt, is flowing into a well-stirred tank. At the same time, the mixture in the tank is flowing out. The goal is to determine how much salt is in the tank as a function of t.

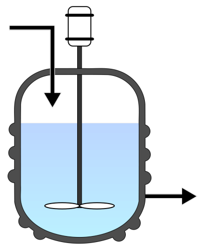

Figure 2.2. *Schematic of a continuous stirred tank.*

The quantities of interest in this problem are:

$Q(t)$**:** This is the amount of salt in the tank at time t. If the volume of water in the tank is V, and c is the concentration of salt in the water, then $Q = cV$.

R_{in}**:** This is the rate that salt is flowing into the tank. If the incoming volumetric flow rate is F_{in}, and c_{in} is the concentration of salt in the incoming water, then $R_{in} = c_{in}F_{in}$.

R_{out}**:** This is the rate that salt is flowing out of the tank. If the outgoing volumetric flow rate is F_{out}, then $R_{out} = cF_{out}$.

If the initial amount of salt in the tank is Q_0, then the resulting IVP for Q is:

$$\begin{aligned} \frac{dQ}{dt} &= R_{in} - R_{out}, \\ Q(0) &= Q_0. \end{aligned}$$

Example 1

Suppose that salt water, containing 1/2 lbs of salt per gal, is poured into a tank at 2 gal/min. Also, the water flows out of the tank at the same rate. If the tank starts out with 100 gal of water, with 10 lbs of salt per gal, find a formula for the total amount of salt in the tank.

Setup

inflow: Since $F_{in} = 2$, and $c_{in} = 1/2$, then $R_{in} = 1$.

outflow: Since the mixture flows out at 2 gal/min, then the volume of water in the tank stays at 100 gal. Also, since $F_{out} = 2$ and $c = Q/100$, then $R_{out} = \frac{1}{50}Q$.

$t = 0$: Given that at the start there are 10 lbs of salt per gal, $Q(0) = 1000$.

The resulting IVP for Q is:

$$\frac{dQ}{dt} = 1 - \frac{1}{50}Q, \tag{2.32}$$

$$Q(0) = 1000. \tag{2.33}$$

Note that because of the way the variables have been defined, Q is measured in pounds and t is measured in minutes.

Solution

Using separation of variables, or the integrating factor solution (2.22), one finds that $Q(t) = 50 + 950e^{-t/50}$.

Question: What is the eventual concentration of salt in the tank?

Answer using solution: Since $\lim_{t\to\infty} Q(t) = 50$, then the eventual concentration is $50/V = \frac{1}{2}$ lbs/gal.

Answer using physical reasoning: The concentration in the tank will eventually be the same as the concentration for the incoming flow, and so the answer is $\frac{1}{2}$ lbs/gal.

Answer using math reasoning: It is possible to determine the eventual concentration directly from the differential equation, without knowing the solution. How this is done is explained in Section 2.4 (also, see Exercise 6 in that section). ■

Example 2

Salt water, containing 3 lbs of salt per gal, flows into a 50 gal drum at 2 gal/sec. If the drum initially contains 10 gal of pure water, find a formula for Q as a function of t.

Comments about this problem: There is no outflow, so the volume of water will increase. However, it's a 50 gal drum, so eventually it will fill and start running over. When this occurs there is outflow, at a rate equal to the incoming rate. To account for this, the problem needs to be split into two phases, one where the volume is increasing, and the second when it is a constant.

Solution

Phase 1: In this case, $R_{in} = 6$, $R_{out} = 0$, and $Q(0) = 0$. The resulting IVP is

$$\frac{dQ}{dt} = 6$$

$$Q(0) = 0$$

The solution is $Q(t) = 6t$. Also, the volume of water in the tank is $V = 10 + 2t$. So, this solution for Q holds for $V \le 50$, which means that $t \le 20$.

Phase 2: As before, $R_{in} = 6$. For the outflow, the rate is 2 gal/sec and the concentration in the outflow is $Q/50$. This means that $R_{out} = Q/25$. Now, this phase starts at $t = 20$, and the amount of salt in the tank at the start is 120 (this comes from the solution for Phase 1). This means that the problem to solve is

$$\begin{aligned} \frac{dQ}{dt} &= 6 - \frac{1}{25}Q, \qquad \text{for} \;\; 20 < t, \\ Q(20) &= 120. \end{aligned}$$

What is different about this problem is the time interval, which is not the usual $0 \le t$. However, this does not interfere with our solution methods, and the solution can be found using an integrating factor or separation of variables. One finds that the general solution of the differential equation is

$$Q(t) = 150 + Ae^{-t/25}.$$

From the requirement that $Q(20) = 120$ it follows that $A = -30e^{4/5}$.

The Solution: Combining the Phase 1 and Phase 2 solutions, we get

$$Q(t) = \begin{cases} 6t & \text{if} \quad 0 \le t \le 20, \\ 150 - 30e^{(20-t)/25} & \text{if} \quad 20 < t. \end{cases} \quad \blacksquare$$

2.3.2 ▪ Newton's Second Law

Suppose an object with mass m is moving along the x-axis. Letting $x(t)$ be its position, then its velocity is $v = \frac{dx}{dt}$, and its acceleration is $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$. If the object is acted on by a force F, then from Newton's second law, which states that $F = ma$, we have that

$$m\frac{dv}{dt} = F. \tag{2.34}$$

What sort of differential equation this might be depends on how F depends on v. Once (2.34) is solved for v, then the position is determined by integrating the equation

$$\frac{dx}{dt} = v. \tag{2.35}$$

Typically, the initial velocity $v(0)$ and initial position $x(0)$ are given, and these are used to determine the integration constants obtained when solving the problem.

Vertical Motion

The object is assumed to be moving vertically, either up or down (see Figure 2.3). In this case, $x(t)$ is the distance of the object from the ground. It is also assumed that it is acted on by gravity, F_g, and a drag force, F_d. Consequently, the total force is $F = F_g + F_d$. As for what these forces are:

Gravitational force: Assuming the gravitational field is uniform, then $F_g = -mg$, where g is the gravitational acceleration constant. The minus sign is because the force is in the downward direction.

Drag force: As long as the object is not moving very fast, the drag is proportional to the velocity (see Exercise 10). In this case, $F_d = -cv$, where c is a positive constant. The minus sign is because the force is in the opposite direction to the direction of motion (so, F_d points upward if the object is falling).

Units and Values: In the exercises, the value to use for g is usually stated. If it is not given, then you should leave g unevaluated. Whatever value is used, it is only approximate. If a more physically realistic value is needed, then you should probably use the Somigliana equation. Finally, weight is a force, so for an object that weighs w lbs, its mass can be determined from the equation $w = mg$.

Example: Suppose a ball with a mass of 2 kg is dropped, from rest, from a height of 1000 m. Assume that the forces acting on the object are gravity, and a drag force due to air resistance, with $c = \frac{1}{2}$ kg/s. Assume that $g = 10\,\mathrm{m/s^2}$.

Question 1: What is the resulting IVP for v, and what problem must be solved to find x?

Answer: Since $F = F_g + F_d = -mg - cv$, where $m = 2$ and $c = 1/2$, then from (2.34) the differential equation is

$$\frac{dv}{dt} = -10 - \frac{1}{4}v. \tag{2.36}$$

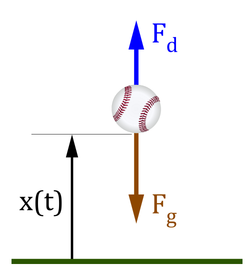

Figure 2.3. *Forces on a falling object.*

Since the object is dropped from rest, then the initial condition is $v(0) = 0$. Once v is known, then x is found by integrating (2.35), and using the fact that $x(0) = 1000$. Also, note that v is measured in meters per second, t is measured in seconds, and x in meters.

Question 2: What is the solution of the IVP, and the resulting solution for x?
Answer: Using the integrating factor solution (2.21), it is found that the general solution is $v = -40 + b_1 e^{-t/4}$, where b_1 is the integration constant. Applying the initial condition we get that

$$v = 40(-1 + e^{-t/4}). \tag{2.37}$$

Integrating $x' = 40(-1 + e^{-t/4})$, yields $x = 40(-t - 4e^{-t/4}) + b_2$. Since $x(0) = 1000$, then $b_2 = 1160$. So, $x = 40(-t - 4e^{-t/4}) + 1160$.

Question 3: What is the terminal velocity v_T of the object?
Answer: The terminal velocity is defined as

$$v_T = \lim_{t \to \infty} v(t).$$

Consequently, from (2.37), we get that $v_T = -40\,\text{m/s}$. It is also possible to determine v_T without solving the IVP, and how this is done is explained in Section 2.4.

Question 4: When does the object hit the ground?
Answer: It hits the ground when $x = 0$, which means that it is the value of t that satisfies $t + 4e^{-t/4} = 29$. This can be solved using a computer, but it is possible to obtain an approximate value fairly easily. Assuming it takes several seconds to hit the ground, then the $4e^{-t/4}$ term should be relatively small. For example, at $t = 10$, $4e^{-t/4} \approx 0.3$, and at $t = 20$, $4e^{-t/4} \approx 0.03$. Consequently, as an approximation, we can replace the equation $t + 4e^{-t/4} = 29$ with $t = 29$. In comparison, the numerically computed value is about $28.997\,\text{s}$. ■

2.3.3 ▪ Logistic Growth or Decay

An assumption often made for the growth of the population of a species is that the population grows at a rate proportional to the current population. If $P(t)$ is the population at time t, then this assumption results in the equation $P' = kP$. The solution is $P(t) = P(0)e^{kt}$, which means that there is exponential growth in the population. This is not sustainable in the real world, and it is more realistic to assume that the rate of growth slows down as the population increases. In fact, if the population is very large, the population should decrease instead of increase. A simple model for this is to assume that $k = r\left(1 - \frac{P}{N}\right)$, where r and N are positive constants. The resulting differential equation is

$$\frac{dP}{dt} = r\left(1 - \frac{P}{N}\right)P, \tag{2.38}$$

which is known as the *logistic equation*. This nonlinear equation can be solved using separation of variables, and partial fractions. Doing this, in the case of when $0 < P < N$,

$$\frac{N}{(N-P)P} dP = r dt \tag{2.39}$$

$$\Rightarrow$$

$$\int \left(\frac{1}{P} + \frac{1}{N-P}\right) dP = \int r dt$$

$$\Rightarrow$$

$$\ln \frac{P}{N-P} = rt + c$$

$$\Rightarrow$$

$$\frac{P}{N-P} = e^{rt+c}.$$

From this, we get

$$P = (N-P)\bar{c}e^{rt}, \tag{2.40}$$

where $\bar{c} = e^c$ is a positive constant. Doing the same thing for the case of when $N < P$, one again gets (2.40) except that $\bar{c}$ is a negative constant. Moreover, for the divide by zero case of when $P = 0$, you get (2.40) but $\bar{c} = 0$. In other words, except for when $P = N$, (2.40) holds with the understanding that $\bar{c}$ is an arbitrary constant. Solving (2.40) for P yields

$$P = \frac{N\bar{c}e^{rt}}{1 + \bar{c}e^{rt}}, \tag{2.41}$$

where $\bar{c}$ is an arbitrary constant. If $P(0) = P_0$, and if $P_0 \neq N$, then one finds that $\bar{c} = P_0/(N - P_0)$. When $P(0) = N$, this is a divide by zero situation in (2.39), and the resulting solution is just the constant $P(t) = N$.

The solution we have derived in (2.41) is known as the *logistic function* or the *logistic curve*. When plotted for $-\infty < t < \infty$ it has a S, or sigmoidal, shape as shown in Figure 2.4. It is one of those functions that appears in so many applications that it deserves its own graph in this textbook (hence Figure 2.4). ■

2.3.4 ▪ Newton's Law of Cooling

The assumption is that the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient

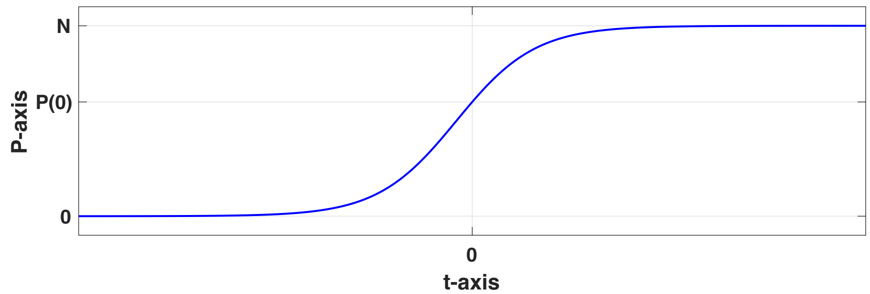

Figure 2.4. *The logistic function (2.41), for $-\infty < t < \infty$, in the case of when $P(0) < N$.*

temperature (i.e., the temperature of its surroundings). This is often referred to as Newton's law of cooling, but it also applies to heating an object.

To write down the mathematical form of this statement, we introduce the following:

$T(t)$**:** This is the temperature of the object at time t.

T_a**:** This is the ambient temperature.

k**:** This is the proportionality coefficient (it is a positive constant).

If the initial temperature of the object is T_0, then the resulting IVP for T is:

$$\frac{dT}{dt} = -k(T - T_a), \tag{2.42}$$

$$T(0) = T_0. \tag{2.43}$$

This problem can be solved using the integrating factor solution (2.22), or by using separation of variables. It is found that $T = T_a + (T_0 - T_a)e^{-kt}$.

Example 1: Cooling a Cup of Coffee

According to the National Coffee Association, the ideal temperature for brewing coffee is 200° F, and to get the most flavor out of it, you should drink it when the coffee is between 120 and 140° F.

Question 1: If the room temperature is 70° F, what is the solution of the resulting IVP for T?

Answer: Since $T_0 = 200$ and $T_a = 70$, then

$$T = 70 + 130e^{-kt}. \tag{2.44}$$

Question 2: If the temperature is 180° F after 2 minutes, determine k.

Answer: From (2.44), $180 = 70 + 130e^{-2k}$. From this one finds that $k = \frac{1}{2}\ln(13/11)\,\frac{1}{\text{min}}$.

Question 3: When should you start drinking the coffee (according to the National Coffee Association)?

Answer: The time when $T = 140$ occurs when $140 = 70 + 130e^{-kt}$, from which one finds that

$$t = 2\frac{\ln(13/7)}{\ln(13/11)}\text{ min.} \tag{2.45}$$

Question 4: What is the computed value for the answer for Question 3?

Answer: It is $t \approx 7.4$ minutes. ■

Example 2: Nonlinear Cooling

Experimentally it has been observed that as certain fluids cool down the change in temperature is proportional to $(T - T_a)^{5/4}$. This is known as the Dulong-Petit law of cooling, and the resulting differential equation is

$$\frac{dT}{dt} = -k(T - T_a)^{5/4}.$$

This requires cooling, and so it requires $T(0) > T_a$.

Question: As in Example 1, suppose that the room temperature is 70° F and $T(0) = 200°$ F. What is the solution of the resulting IVP?

Answer: Separating variables,

$$-\frac{dT}{(T-70)^{5/4}} = kdt.$$

Integrating this equation and then solving for T,

$$\frac{4}{(T-70)^{1/4}} = kt + c$$

$$\Rightarrow$$

$$(T-70)^{1/4} = \frac{4}{kt+c}$$

$$\Rightarrow$$

$$T = 70 + \left(\frac{4}{kt+c}\right)^4.$$

Since $T(0) = 200$, then the above equation gives us that $130 = (4/c)^4$. Solving this we obtain $c = 4/130^{1/4}$. ■

Reality Check: The models that are considered here are used to illustrate how, and where, differential equations arise. As with all models, simplifying assumptions are made to obtain the resulting mathematical problem. Many of these assumptions are not considered or accounted for in our examples, and the same is true for the exercises. As a case in point, Newton's Law of Cooling is usually limited to cases of when $|T - T_a|$ is not very large, and its applicability depends on whether the heat flow is due to conduction, convection, or radiation. Said another way, if you want to impress your family at Thanksgiving by using the solution of the cooking a turkey exercise (Exercise 15), just make sure to check on the turkey temperature regularly to make sure your predictions are correct.

Exercises

In answering the following questions, do not numerically evaluate numbers such as $\sqrt{2}$, $\pi/3$, e^2, $\ln(4/3)$, etc. The exception to this is when the question explicitly asks you to *compute* the answer.

Radioactive Decay

1. The IVP for radioactive decay was derived in Example 1, on page 1.
   a) What is the solution of the IVP for a radioactive material?
   b) If 12 mg of a radioactive material decays to 9 mg in one day, find k.
   c) The half-life of a radioactive material is the time required for it to reach one-half of the original amount. What is the half-life of the material in part (b)?
2. Radiocarbon dating uses the decay of carbon-14 to estimate how long ago something died. The assumption is that the amount of carbon-14 satisfies the radioactive decay problem derived in Example 1, on page 1.
   a) What is the solution of the IVP for a radioactive material?
   b) The half-life of a radioactive material is the time required for it to reach one-half of the original amount. The half-life of carbon-14 is 5,730 years. Use this to determine k.
   c) The amount of carbon-14 is the same in all living organisms. When an organism dies the amount starts to undergo radioactive decay. So, for radioactive dating you know N_0, as well as the current value of N. Explain how knowing N_0, N, and k can be used to determine t (which is the time that has passed since the organism died).
   d) Measurements in 1991 determined that the amount of carbon-14 in the Temple Scroll, which is one of the Dead Sea scrolls found at Qumran, to be 186.18. The amount in living organisms is 238.

Determine (i.e., compute) what two years the scroll could have been written in. Note that in the BC/AD system there is no year zero, so it goes from 1 BC to 1 AD.
Comments: In this problem, the amount of carbon-14 refers to the amount relative to carbon-12. Also, the organism is the parchment from the scroll, and the testing is described in Bonani et al. [1992].

Mixing

3. A tank contains 100 L of salt water with a concentration of 2 g/L. To flush the salt out, pure water is poured in at 4 L/min, and the mixture in the tank flows out at the same rate.
   a) What is the resulting IVP for the total amount $Q(t)$ of salt in the tank?
   b) Solve the IVP determined in part (a).
   c) How long does it take until the amount of salt in the tank is 1% of its original amount?
4. A tank contains 20 L of fresh water. Suppose water, containing $\frac{1}{4}$ g/L of salt, starts to flow into the tank at 2 L/min, and the well-stirred mixture flows out at the same rate.
   a) What is the resulting IVP for the amount $Q(t)$ of salt in the tank?
   b) Solve the IVP determined in part (a).
   c) How much salt is in the tank after one hour?
5. Ten years ago, a factory started operation in a pristine valley. The valley's volume is $10^6\,\text{m}^3$. Each year the factory releases $10^5\,\text{m}^3$ of exhaust through its smoke stacks, and this exhaust contains 1000 kg of pollutants. Assume that the well-mixed polluted air leaves the valley at $10^5\,\text{m}^3/\text{yr}$.
   a) What is the IVP for the amount of pollutant in the valley?
   b) How much pollutant is in the valley now?
6. A small lake contains 60,000 gal of pure water. There is an inlet stream of pure water into the lake, as well as an outlet stream, both flowing at a rate of 100 gal/min. Suppose someone starts pouring water into the lake at the rate of 10 gal/min that contains 5 lbs/gal of a chemical, and they do this for 8 hours. While this happens the inlet stream of pure water is unchanged, and the outflow rate from the lake remains at 100 gal/min.
   a) What is the formula for the volume of the lake while the person is pouring?
   b) For $t < 8$, what IVP must be solved to determine the amount of the chemical in the lake?

c) How much of the chemical is in the lake when the person stops pouring?

d) Once the person stops pouring, what IVP must be solved to determine how much of the chemical is in the lake?

Newton's Second Law

7. A mass of 10 kg is shot upward from the surface of the Earth with a velocity of 100 m/s. In addition to gravity, assume that there is a drag force $F_d = -cv$, where $c = 5\,\text{kg/s}$. Assume that $g = 10\,\text{m/s}^2$.

   a) Write down the IVP for v, and then find its solution.

   b) Find x.

   c) How high does the object get?

8. A skydiver weighing 176 lbs drops from a plane that is at an altitude of 5000 ft. Assume that $g = 32\,\text{ft/s}^2$.

   a) Before the parachute opens, the forces on the skydiver are gravity and a drag force $F_d = -cv$. Assuming $v(0) = 0$, write down the IVP for v, and then find the solution.

   b) It is claimed that the terminal velocity of a person falling is -120 mph. Use this to determine c.

   c) If the parachute is opened after 10 s of free fall, what is the speed of the skydiver when it opens?

   d) Find the distance the skydiver falls before the parachute opens.

   e) When the parachute is open, the drag force increases by a factor of 8 from the free fall drag force. What is the resulting terminal velocity of the skydiver?

9. A spherical object sinking to the bottom of a lake is acted on by three forces: a drag force $F_d = -cv$, a buoyant force F_b, and gravity F_g. According to Archimedes' principle, the buoyant force is equal to the weight of the water that is displaced by the sphere.

   a) What is the formula for F_b in terms of the sphere's radius a, the water density ρ, and g?

   b) The differential equation for the velocity of the sphere has the form $mv' = A - cv$. What is A?

   c) Assuming the sphere is released from rest, solve the resulting IVP for v.

   d) Find a formula for the terminal velocity in terms of c, a, ρ, and g. What condition must be satisfied if the sphere is sinking?

   e) Assume the object is released a distance L from the bottom of the lake. Also assume that it takes a while for it to hit the bottom. Use an approximation similar to the one used in Question 4 on page 24 to derive an approximate formula for the time it takes it to hit the bottom.

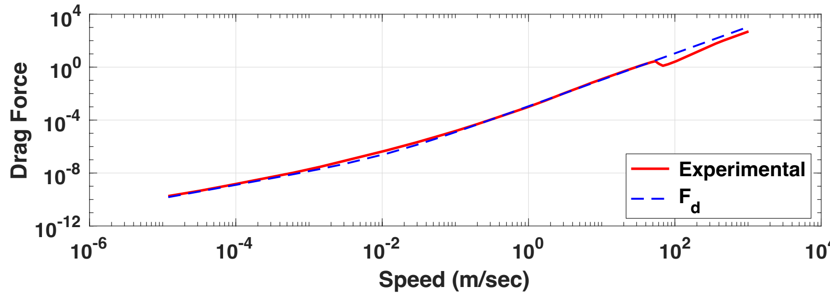

Figure 2.5. *Drag force on a smooth sphere as a function of the speed [Roos and Willmarth, 1971, NASA, 2025]. The function F_d is used in Exercise 10.*

10. A spherical object falling in the atmosphere is acted on by gravity, F_g, and a drag force F_d. It is assumed that $F_d = -cv(1 - \beta v)$, where v is the velocity. Both c and β are positive constants.
    a) Assuming the sphere is dropped from rest, what is the resulting IVP for v?
    b) Solve the IVP for v.
    c) Find a formula for the terminal velocity in terms of m, c, β, and g.
    d) The constants in F_d are $c = 6\pi R\mu$ and $\beta = R\rho/(9\pi\mu)$, where R is the radius of the sphere, ρ is the air density, and μ is the air viscosity. For a baseball falling in the atmosphere, $R = 0.037$, $\mu = 1.8 \times 10^{-5}$, and $\rho = 1.2$ (using kg, m, s units). Also, $m = 0.14$ and assume that $g = 9.8$. Compute the terminal velocity. How does this compare to what is the speed of a typical fastball in professional baseball?
    Comment: The drag force used in this problem is close to what is observed experimentally. To demonstrate this, the experimentally determined values of the drag, and the values determined using F_d, are shown in Figure 2.5 as a function of the speed $|v|$. This data also shows that the assumption $F_d = -cv$ is only valid if the speed is no more than about 10^{-2} m/s.

Logistic Growth or Decay

11. It is often found that a population will grow exponentially if the population is very small, and it will decrease exponentially if the population is very large. A model for this is due to Beverton and Holt, and the equation to solve is
$$P' = r\frac{1 - \frac{P}{N}}{1 + \frac{P}{N}}P,$$
where r and N are positive constants.

a) Assuming that $P(0) = \frac{1}{2}N$, solve the resulting IVP for P.

b) What is the limiting population $P(\infty) = \lim_{t\to\infty} P(t)$?

12. The population of fish in a large lake can be modeled using the logistic equation. If, in addition, the fish are caught at a constant rate h, the equation for the population becomes

$$P' = r\left(1 - \frac{P}{N}\right)P - h,$$

where r and N are positive constants. In this problem take $r = 4$, $h = 750$, and $N = 1000$. Also, $P(0) = 1000$.

a) Solve the IVP for P.

b) What is the limiting population? In other words, what is $P(\infty) = \lim_{t\to\infty} P(t)$?

Cooling or Heating

13. Suppose coffee has a temperature of 200° F when freshly poured, and the room temperature is 72° F. In this exercise use Newton's law of cooling.

a) What IVP does the temperature of the coffee satisfy?

b) What is the solution of the IVP?

c) If the coffee cools to 136° F in five minutes, what is k?

d) When does the coffee reach a temperature of 150° F?

14. Redo the previous exercise but use the Dulong-Petit law of cooling.

15. To cook a turkey you are to put it into a 350°F oven, and cook it until it reaches 165°F. In answering the following questions, assume Newton's law of cooling is used.

a) Suppose the turkey starts out at room temperature, which is 70°F. What IVP does the temperature satisfy?

b) Suppose that after two hours in the oven, the temperature of the turkey is 140°F. How much longer before it is done?

c) Suppose the turkey is taken from the refrigerator, which is set to 40°F, and put directly into the oven. How much longer does it take to cook than when the turkey starts out at room temperature? The value for k is the same as in part (b).

16. A homicide victim was discovered at 1 p.m. in a room that is kept at 70°F. When discovered, the temperature of the body was 90°F, and one hour later it had dropped to 85°F.

a) Assuming Newton's Law of Cooling, and normal body temperature is 98.6°F, how long had the person been dead when the body was discovered?

b) Compute the time of death. Round your answer so it just gives the hour and minute (e.g., 7:13 a.m. or 5:32 p.m.).

17. The idea underlying Newton's law is that the rate of cooling, or heating, depends on the temperature difference $T - T_a$. More specifically, $T' = f(T - T_a)$ where, to be consistent with the laws of heat transfer, $f(0) = 0$ and $f'(0) < 0$. Now, using Taylor's theorem, $f(s) = f(0)+f'(0)s+\frac{1}{2}f''(0)s^2+\frac{1}{6}f'''(0)s^2+\cdots$. Newton's law of cooling is based on the linear approximation $f(s) \approx f(0) + f'(0)s = -ks$ where $k > 0$. This yields the differential equation (2.42). In this problem we consider the somewhat better quadratic approximation $f(s) \approx f(0) + f'(0)s + \frac{1}{2}f''(0)s^2 = -ks - Ks^2$ where $k > 0$. You can assume that $K > 0$.

a) If $f(s) = -ks-Ks^2$, then what is the resulting differential equation for T?

b) To find T it makes things easier to introduce the variable $S(t) = T(t) - T_a$. Rewrite the differential equation in part (a) in terms of S. Also, if $T(0) = T_0$, what is $S(0)$?

c) Solve the resulting IVP in part (b) for S, and then use this to show that

$$T = T_a + \frac{kce^{-kt}}{1 - Kce^{-kt}} ,$$

where $c = S_0/(k + KS_0)$ and $S_0 = T_0 - T_a$.

d) Using (2.44), it was found that you should wait about 7.4 minutes to drink the coffee. Taking $k = \frac{1}{2}\ln(13/11)$ and $K = 0.01$, compute how long you need to wait using the solution for T from part (c).

2.4 ▪ Steady States and Stability

All of the applications considered in the previous section have one thing in common: the solution eventually approaches a constant, or steady state, value. This is not unusual, as this is what often happens. What is of interest here is whether it is possible to determine the eventual steady state without actually having to solve the problem.

To illustrate, as explained in the previous section, the population $P(t)$ of a species is determined by solving

$$P' = f(P), \tag{2.46}$$

where, for this example, we will take

$$f(P) = 2(3 - P)P. \tag{2.47}$$

The solution of this equation is given in (2.41), and it is plotted in Figure 2.6 for the case of when $P(0) = 0.1$, and when $P(0) = 4.5$. It shows that